Pointwise Partial Information Decomposition Using Specificity and Ambiguity Lattices

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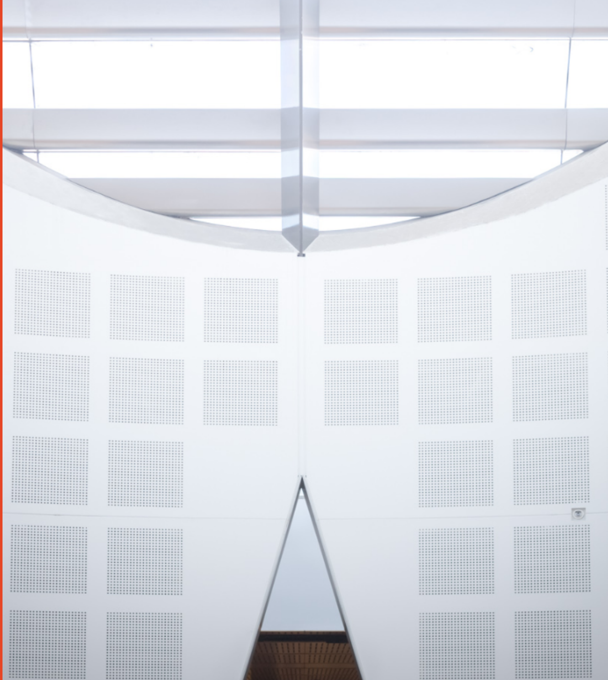


Pointwise Information Theory

- (Shannon) entropy: expected information in a realisation of a random variable

$$H(X) = \sum_x p(x) \log \tfrac{1}{p(x)} \geq 0$$

- Mutual information (MI) quantifies the expected information provided by Y about X, or vice versa

$$I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

- From four postulates, Fano (1961) derived the **pointwise** mutual information which quantifies the information provided by event y about the event x, or vice versa

$$i(x;y) = \log \frac{p(x,y)}{p(x)p(y)}$$

- Corollaries: (average) mutual information, pointwise entropy and (average) entropy

Pointwise mutual information can be negative!

Unique, Redundant and Synergistic Information

Consider three random variables S_1, S_2 and T and suppose we are interested in predicting the value of T from S_1 and S_2

- **Unique information** $U(T : S_1 \setminus S_2)$
  Source S_1 may contain information about T that source S_2 does not (or vice versa)

p	s_1	s_2	t
¼	0	0	0
¼	0	1	0
¼	1	0	1
¼	1	1	1

- **Redundant information** $R(T : S_1, S_2)$
  Source S_2 may contain the same information as source S_2 about T

p	s_1	s_2	t
0	0	0	1/2
1	1	1	1/2

- **Synergistic information** $C(T : S_1, S_2)$
  It is possible that neither source Z nor source Y contain information about X, yet take together they do; e.g. XOR

t	s_1	s_2	t
¼	0	0	0
¼	0	1	1
¼	1	0	1
¼	1	1	0

Information Decomposition

In general, unique, redundant and synergistic information are present simultaneously

p	s_1	s_2	t
1/4	0	0	0
1/4	0	1	1
1/4	1	0	1
1/4	1	1	1

- Mutual information captures

$$I(T;S_1) = R(T:S_1,S_2) + U(T:S_1\backslash S_2)$$
$$I(T;S_2) = R(T:S_1,S_2) + U(T:S_2\backslash S_1)$$

- Joint mutual information captures

$$I(T;S_1S_2) = R(T:S_1,S_2) + U(T:S_1\backslash S_2) + U(T:S_2\backslash S_1) + C(T:S_1,S_2)$$

- Three equations with four unknowns
  - Define one of the unique, redundant, or synergistic information and solve

- We would like to generalise these notions to any number of variables

Partial Information Decomposition

- PID of Williams and Beer (2010) provides an axiomatic framework for extending information decomposition to arbitrary number of source variables

Axioms (PID)

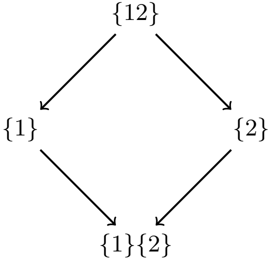
(1) *Symmetry*: $R(T : S_1, \ldots, S_n)$ is invariant under permutations of the S_i's

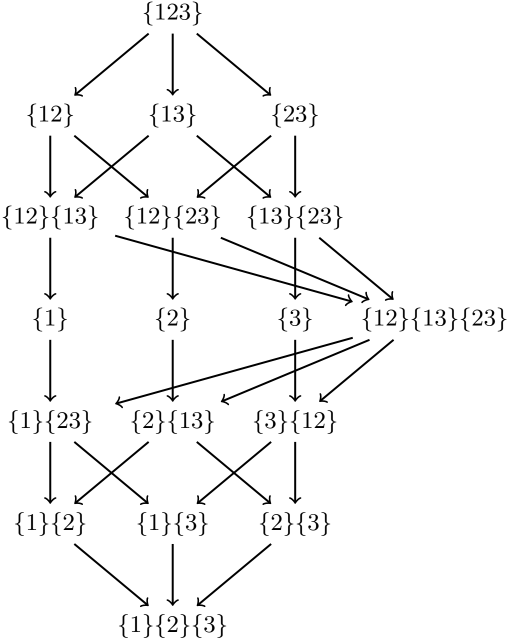
(2) *Self-redundancy*: $R(T : S_i) = I(T; S_i)$

(3) *Monotonicity*: $R(T : S_1, \ldots, S_n) \leq R(T : S_1; \ldots; S_{n-1})$

- Based upon the idea that redundancy is in some way analogous to set intersection
- Yields a structure for multivariate information called the **redundancy lattice**
- No well-accepted definition of unique, redundant, or synergistic information which is compatible with PID in general has emerged

Redundancy lattice





Pointwise Information Decomposition

- In principle, we should be able to decompose pointwise information for each realisation

$$
\begin{aligned}
i(t;s_1) &= r(t:s_1,s_2) + u(t:s_1\backslash s_2)\\
i(t;s_2) &= r(t:s_1,s_2) + u(t:s_2\backslash s_1)\\
i(t;s_1 s_2) &= r(t:s_1,s_2) + u(t:s_1\backslash s_2) + u(t:s_2\backslash s_1) + c(t:s_1,s_2)
\end{aligned}
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- These should average to yield the (average) information decomposition

$$\begin{aligned} R(T:S_1,S_2) &= \big\langle r(t:s_1,s_2)\big\rangle & \qquad U(T:S_1\backslash S_2) &= \big\langle u(t:s_1\backslash s_2)\big\rangle \\ C(T:S_1,S_2) &= \big\langle c(t:s_1,s_2)\big\rangle & \qquad U(T:S_2\backslash S_1) &= \big\langle u(t:s_2\backslash s_1)\big\rangle \end{aligned}$$

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R(T : S_1, S_2) &= \langle r(t : s_1, s_2) \rangle & \qquad U(T : S_1 \backslash S_2) &= \langle u(t : s_1 \backslash s_2) \rangle \\
C(T : S_1, S_2) &= \langle c(t : s_1, s_2) \rangle & \qquad U(T : S_2 \backslash S_1) &= \langle u(t : s_2 \backslash s_1) \rangle
\end{aligned}
$$

- And we should have the usual PID for the (average) informations

$$
\begin{aligned}
I(T; S_1) &= R(T : S_1, S_2) + U(T : S_1 \backslash S_2) \\
I(T; S_2) &= R(T : S_1, S_2) + U(T : S_2 \backslash S_1) \\
I(T; S_1 S_2) &= R(T : S_1, S_2) + U(T : S_1 \backslash S_2) + U(T : S_2 \backslash S_1) + C(T : S_1, S_2)
\end{aligned}
$$

Example: PwUnq

- Consider the example called Pointwise Unique (PwUnq) from Finn et al. (2017b)

p	s_1	s_2	t	
1/4	0	1	1	
1/4	1	0	1	
1/4	0	2	2	
1/4	2	0	2	
Expected values				

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p	s_1	s_2	t	$i(t; s_1)$	$i(t; s_2)$	$i(t; s_1 s_2)$	
1/4	0	1	1	0	1	1	
1/4	1	0	1	1	0	1	
1/4	0	2	2	0	1	1	
1/4	2	0	2	1	0	1	
Expected values				1/2	1/2	1	

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p	s_1	s_2	t	$i(t; s_1)$	$i(t; s_2)$	$i(t; s_1 s_2)$	r
1/4	0	1	1	0	1	1	0
1/4	1	0	1	1	0	1	0
1/4	0	2	2	0	1	1	0
1/4	2	0	2	1	0	1	0
Expected values				1/2	1/2	1	0

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1/4	0	1	1	0	1	1	0	0	1	0
1/4	1	0	1	1	0	1	0	1	0	0
1/4	0	2	2	0	1	1	0	0	1	0
1/4	2	0	2	1	0	1	0	1	0	0
Expected values				1/2	1/2	1	0	1/2	1/2	0

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1/4	1	0	1	1	0	1	0	1	0	0
1/4	0	2	2	0	1	1	0	0	1	0
1/4	2	0	2	1	0	1	0	1	0	0
Expected values				1/2	1/2	1	0	1/2	1/2	0

- According to Williams and Beer (2010), Bertschinger et al. (2014), Griffith and Koch (2014) and Harder et al. (2013)

$$R = \langle r \rangle = 1/2 \text{ bit}$$

Pointwise Partial Information Decomposition

- Idea: rewrite PID axioms using pointwise MI instead of (average) MI

Axioms (PPID)

(1) *Symmetry*: $r(t : s_1, \ldots, s_n)$ is invariant under permutations of the s_i's

(2) *Self-redundancy*: $r(t : s_i) = i(t; s_i)$

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Problem: pointwise mutual information is not non-negative

- How do we deal with this issue?

Specificity and Ambiguity

- Idea: split the pointwise mutual information into non-negative entropic components

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- In Finn et al. (2017a), we proved pointwise information provided by s about t must be split in the following way

$$i(s\to t) = i^{+}(s\to t) - i^{-}(s\to t)$$

where

$$\text{(Specificity)} \qquad i^{+}(s\to t) = h(s) \qquad\qquad i^{-}(s\to t) = h(s|t) \qquad \text{(Ambiguity)}$$

Specificity and Ambiguity

- Idea: split the pointwise mutual information into non-negative entropic components
- In Finn et al. (2017a), we proved pointwise information provided by s about t must be split in the following way

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where

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Axioms (PPID using Specificity and Ambiguity)

(1) *Symmetry*: $r^{\pm}(t : s_1, \ldots, s_n)$ is invariant under permutations of the s_i's
(2) *Self-redundancy*: $r^{\pm}(t : s_i) = i^{\pm}(t; s_i)$
(3) *Monotonicity*: $r^{\pm}(t : s_1, \ldots, s_n) \leq r^{\pm}(t : s_1; \ldots; s_{n-1})$

- Yields two lattices (per realisation): the specificity and ambiguity lattices

Redundancy Measure on the Lattices

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- We define the redundant specificity and redundant ambiguity to be

$$r^{+}_{\text{min}}(s_1, \ldots, s_k \to t) = \min_{s_j} h(s_j) \qquad r^{-}_{\text{min}}(s_1, \ldots, s_k \to t) = \min_{s_j} h(s_j|t)$$

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- There is another axiom (Axiom 4) which helps justify this definition
  - Operational justification in terms of Kelly gambling

Example: PwUnq

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/4	0	1	1	1	1	2	1	2	1	1	0	1	0	1	0	0	0
1/4	1	0	1	2	1	1	1	2	1	1	1	0	0	1	0	0	0
1/4	0	2	2	1	1	2	1	2	1	1	0	1	0	1	0	0	0
1/4	2	0	2	2	1	1	1	2	1	1	1	0	0	1	0	0	0
Expected values				3/2	1	3/2	1	2	1	1	1/2	1/2	0	1	0	0	0

Example: PwUnq

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/4	0	1	1	1	1	2	1	2	1	1	0	1	0	1	0	0	0
1/4	1	0	1	2	1	1	1	2	1	1	1	0	0	1	0	0	0
1/4	0	2	2	1	1	2	1	2	1	1	0	1	0	1	0	0	0
1/4	2	0	2	2	1	1	1	2	1	1	1	0	0	1	0	0	0
Expected values				3/2	1	3/2	1	2	1	1	1/2	1/2	0	1	0	0	0

- Recombining the average specificities and average ambiguities yields the PID

$$R(T : S_1, S_2) = 1 - 1 = 0 \text{ bit} \qquad U(T : S_1 \backslash S_2) = 1/2 - 0 = 1/2 \text{ bit}$$

$$C(T : S_1, S_2) = 0 - 0 = 0 \text{ bit} \qquad U(T : S_2 \backslash S_1) = 1/2 - 0 = 1/2 \text{ bit}$$

- Matches the PPID suggested earlier

Example: XOR

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/4	0	0	0	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	0	1	1	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	1	0	1	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	1	1	0	1	1	1	1	2	1	1	0	0	1	1	0	0	0
Expected values				1	1	1	1	2	1	1	0	0	1	1	0	0	0

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1/4	0	0	0	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	0	1	1	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	1	0	1	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	1	1	0	1	1	1	1	2	1	1	0	0	1	1	0	0	0
Expected values				1	1	1	1	2	1	1	0	0	1	1	0	0	0

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- Identifies redundancy due to shared knowledge from Bertschinger et al. (2013)

Example: IMPRDN

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
½	0	0	0	1	0	$\lg 8/5$	0	1	0	$\lg 8/5$	$\lg 5/4$	0	0	0	0	0	0
⅜	1	1	1	1	0	$\lg 8/3$	$\lg 4/3$	$\lg 8/3$	$\lg 4/3$	1	0	$\lg 4/3$	0	0	0	$\lg 4/3$	0
⅛	1	0	1	1	0	$\lg 8/5$	2	3	2	$\lg 8/5$	$\lg 5/4$	0	2	0	0	2	0
Expected				1	0	0.954	0.406	1.406	0.406	0.799	0.201	0.156	0.250	0	0	0.406	0

- Recombining the average specificities and average ambiguities yields the PID

$R(T : S_1, S_2) = 0.799 - 0 = 0.799$ bit $\quad U(T : S_1 \backslash S_2) = 0.201 - 0 = 0.201$ bit

$C(T : S_1, S_2) = 0.25 - 0.25 = 1$ bit $\quad U(T : S_2 \backslash S_1) = 0.156 - 0.406 = -0.25$ bit

Example: ImpRdn

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/2	0	0	0	1	0	$\lg 8/5$	0	1	0	$\lg 8/5$	$\lg 5/4$	0	0	0	0	0	0
3/8	1	1	1	1	0	$\lg 8/3$	$\lg 4/3$	$\lg 8/3$	$\lg 4/3$	1	0	$\lg 4/3$	0	0	0	$\lg 4/3$	0
1/8	1	0	1	1	0	$\lg 8/5$	2	3	2	$\lg 8/5$	$\lg 5/4$	0	2	0	0	2	0
Expected				1	0	0.954	0.406	1.406	0.406	0.799	0.201	0.156	0.250	0	0	0.406	0

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$$C(T : S_1, S_2) = 0.25 - 0.25 = 1 \text{ bit} \qquad U(T : S_2 \backslash S_1) = 0.156 - 0.406 = -0.25 \text{ bit}$$

- May be negative unique information on average if a source is uniquely misinformative

Comparison to Other Decompositions and Measures

- Approach is most similar to Ince (2017) but differs in how the non-positivity is dealt with

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- This also makes it similar to S_{VK} of Griffith and Koch (2014)
- Like other measures, there is no target monotonicity, i.e. don not have that

$$R_{\mathsf{min}}\big(S_1, S_2 \to T_1\big) \leq R_{\mathsf{min}}\big(S_1, S_2 \to T_1, T_2\big)$$

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- Like other measures, there is no target monotonicity, i.e. don not have that
$$R_{\text{min}}\big(S_1, S_2 \to T_1\big) \leq R_{\text{min}}\big(S_1, S_2 \to T_1, T_2\big)$$
- But unlike other measures, there is a target chain rule
$$R_{\text{min}}\big(S_1, S_2 \to T_1, T_2\big) = R_{\text{min}}\big(S_1, S_2 \to T_1\big) + R_{\text{min}}\big(S_1, S_2 \to T_2 | T_1\big)$$

Example: TwoBitCopy

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/4	0	0	00	1	0	1	0	2	0	1	0	0	1	0	0	0	0
1/4	0	1	01	1	0	1	0	2	0	1	0	0	1	0	0	0	0
1/4	1	0	10	1	0	1	0	2	0	1	0	0	1	0	0	0	0
1/4	1	1	11	1	0	1	0	2	0	1	0	0	1	0	0	0	0
Expected values				1	0	1	0	2	0	1	0	0	1	0	0	0	0

- Recombining the average specificities and average ambiguities yields the PID

$$R(T : S_1, S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_1 \setminus S_2) = 0 - 0 = 0 \text{ bit}$$

$$C(T : S_1, S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_2 \setminus S_1) = 0 - 0 = 0 \text{ bit}$$

- Result is the same as it is for I_{min} of Williams and Beer (2010)

Example: TwoBitCopy

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/4	0	0	00	1	0	1	0	2	0	1	0	0	1	0	0	0	0
1/4	0	1	01	1	0	1	0	2	0	1	0	0	1	0	0	0	0
1/4	1	0	10	1	0	1	0	2	0	1	0	0	1	0	0	0	0
1/4	1	1	11	1	0	1	0	2	0	1	0	0	1	0	0	0	0
Expected values				1	0	1	0	2	0	1	0	0	1	0	0	0	0

- Recombining the average specificities and average ambiguities yields the PID

$$R(T : S_1, S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_1 \setminus S_2) = 0 - 0 = 0 \text{ bit}$$

$$C(T : S_1, S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_2 \setminus S_1) = 0 - 0 = 0 \text{ bit}$$

- Result is the same as it is for I_{min} of Williams and Beer (2010)
- The measure and decomposition does not possess the identity property
  - Does mean that we can use this decomposition for more than 3 variables

Example: TwoBitCopy Horse Race

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
¼	b	r	a	1	0	1	0	2	0	1	0	0	1	0	0	0	0
¼	b	g	b	1	0	1	0	2	0	1	0	0	1	0	0	0	0
¼	w	r	c	1	0	1	0	2	0	1	0	0	1	0	0	0	0
¼	w	g	d	1	0	1	0	2	0	1	0	0	1	0	0	0	0
Expected values				1	0	1	0	2	0	1	0	0	1	0	0	0	0

- Recombining the average specificities and average ambiguities yields the PID

$$R(T : S_1, S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_1 \backslash S_2) = 0 - 0 = 0 \text{ bit}$$

$$C(T : S_1, S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_2 \backslash S_1) = 0 - 0 = 0 \text{ bit}$$

- Result is the same as it is for I_{min} of Williams and Beer (2010)
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