Local interpretations of PID

PID workshop, FIAS

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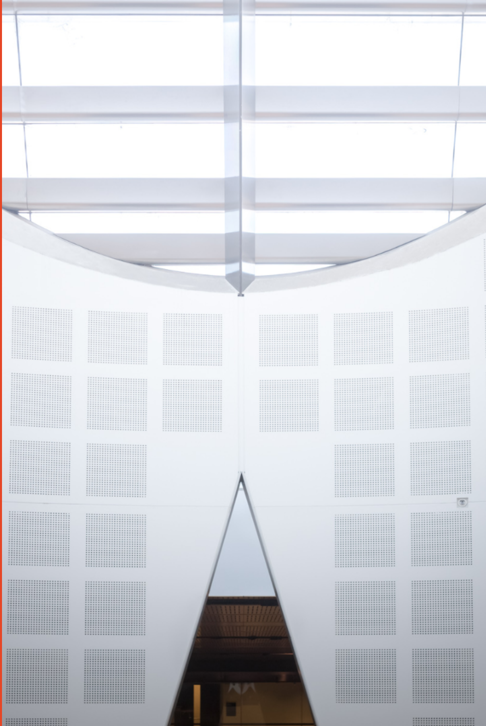
THE UNIVERSITY OF SYDNEY



What do we mean by local information?

- The entropy and mutual information are the expected values over the local or pointwise values:

$$H(X) = \langle h(x) \rangle, \qquad \text{where,} \qquad h(x) = -\log p(x);$$

$$I(X;Y) = \langle i(x;y) \rangle, \qquad \text{where,} \qquad i(x;y) = \log \frac{p(x|y)}{p(x)}.$$

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- Local mutual information can be negative! 

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- Fano — local mutual information can be derived as the primary citizen from four postulates:
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  3. addativity; and
  4. separation for independent ensembles.
- If X and Y are time series, local values measure dynamics over time which would be useful in applications of PID.

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Axiom: *Localizability*

There exists a local measure $i_\cap(t; \mathbf{s}_1, \ldots, \mathbf{s}_k)$ for the redundancy of a specific observation $\{t, \mathbf{s}_1, \ldots, \mathbf{s}_k\}$ of $\{T, \mathbf{S}_1, \ldots, \mathbf{S}_k\}$, such that:

1. $i_\cap(t; \mathbf{s}_1, \ldots, \mathbf{s}_k)$ satisfies the corresponding symmetry and self-redundancy axioms as per $I_\cap(T; \mathbf{S}_1, \ldots, \mathbf{S}_k)$;
2. $I_\cap(T; \mathbf{S}_1, \ldots, \mathbf{S}_k) = \langle i_\cap(t; \mathbf{s}_1, \ldots, \mathbf{s}_k) \rangle$;
3. $i_\cap(t; \mathbf{s}_1, \ldots, \mathbf{s}_k)$ is *once-differentiable* with respect to changes in $p(t, \mathbf{s}_1, \ldots, \mathbf{s}_k)$; and
4. $i_\cap(t; \mathbf{s}_1, \ldots, \mathbf{s}_k)$ is *uniquely defined* for the given candidate redundancy measure.

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Global PID for ensemble

$$\begin{aligned} I(T;S_1) &= U_1 + R \\ I(T;S_2) &= U_2 + R \\ I(T;S_1,S_2) &= U_1 + U_2 + R + S \end{aligned}$$

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Local PID for each configuration

$$i(t;s_1) = u_1^i + r^i$$
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$$U_1 = \langle u_1^i \rangle \qquad U_2 = \langle u_2^i \rangle \qquad R = \langle r^i \rangle \qquad S = \langle s^i \rangle$$

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p	t	s_1	s_2	$i(t;s_1)$	$i(t;s_2)$	$i(t;s_1,s_2)$	u_1	u_2	r	s
$1/4$	0	0	0	$\log 4/3$	$\log 4/3$	$\log 4/3$	u_1^1	u_2^1	r^1	s^1
$1/4$	0	0	1	$\log 4/3$	$\log 2/3$	$\log 4/3$	u_1^2	u_2^2	r^2	s^2
$1/4$	0	1	0	$\log 2/3$	$\log 4/3$	$\log 4/3$	u_1^3	u_2^3	r^3	s^3
$1/4$	1	1	1	1	1	2	u_1^4	u_2^4	r^4	s^4
Expected values				0.311	0.31	0.811	U_1	U_2	R	S

Advantages of the local approach to PID

"*However, [I_{min}] ignore[s] that even though X and Y give the same amount of information about an outcome z, they tell something different about the change of the distribution $p(z)$ after an observation in X or Y has been made.*"

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"*Altogether, we conclude that S_{max} overestimates the intuitive synergy by miscategorizing merely unique information as synergistic whenever two or more predictors have unique information about the target.*"

— Griffith and Koch 2014

Advantages of the local approach to PID

"*Even though* S_1 *and* S_2 *give the same amount of information about an outcome* t*, they tell you something different.*"

Advantages of the local approach to PID

"*Even though S_1 and S_2 give the same amount of information about an outcome t, they tell you something different.*"

- Reduced OR (`ROR`):

p	t	s_1	s_2	$i(t;s_1)$	$i(t;s_2)$	$i(t;s_1,s_2)$
$1/2$	0	0	0	0.415	0.415	1
$1/4$	1	0	1	-0.585	1	1
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Expected values				0.311	0.311	1

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- Unique information may depend on your perspective.

Existing measures and localisation: I_{red}

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$$I_{\text{red}}(T : S_1; S_2) = \min\left\{I_T^{\pi}(S_1 \searrow S_2), I_T^{\pi}(S_2 \searrow S_1)\right\},$$

where

$$I_T^{\pi}(S_1 \searrow S_2) = \sum_{t, s_1} p(t, s_1) \log \frac{p_{(s_1 \searrow S_2)}(t)}{p(t)}$$

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- Defining,
$$r(t : s_1; s_2) = \log \frac{p_{(s_1 \searrow S_2)}(t)}{p(t)},$$
does not work when $I_T^{\pi}(S_1 \searrow S_2) = I_T^{\pi}(S_2 \searrow S_1)$ because
$$\log \frac{p_{(s_1 \searrow S_2)}(t)}{p(t)} = \log \frac{p_{(s_2 \searrow S_1)}(t)}{p(t)},$$
is not generally true—i.e. the choice of $r(t : s_1; s_2)$ is not unique.

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$$\widetilde{UI}(T : S_1 \setminus S_2) = \min_{Q \in \Delta_P} MI_Q(T : S_1 | S_2)$$

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  does not work as Q does not necessarily have the same support as P.

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  does not work as Q does not necessarily have the same support as P. For example, ROR

p	t	s_1	s_2
$1/2$	0	0	0
$1/4$	1	0	1
$1/4$	1	1	0

$$\xrightarrow[\text{to } Q]{P \text{ minimises}}$$

q	t	s_1	s_2
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More on ROR

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1. "*Shared information and unique information should depend only on the marginal distributions of the pairs* (X, Y) *and* (X, Z)*.*"

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2. Say an observer viewing S_1 knows the full joint distribution. If they observe $S_1 = 1$, then they uniquely know the value of the target $T = 1$ in that local case.

More on RoR

The unique information in RoR depends on the chosen perspective

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- The first perspective is common to all of I_{min}, I_{red} and $\widetilde{UI}$. They say there is no unique information in RoR seems to be incompatible with the second perspective.

Another interesting local example

IMPERFECTRDN — Griffith et al. 2014; Ince 2016

p	t	s_1	s_2	$i(t;s_1)$	$i(t;s_2)$	$i(t;s_1,s_2)$	u_1	u_2	r	s
0.5	0	0	0	1	$\log(5/3)$	1				
0.1	1	1	0	1	$\log(1/3)$	1				
0.4	1	1	1	1	1	1				
Expected values				1	0.610	1				

Assumptions:

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ImperfectRdn — Griffith et al. 2014; Ince 2016

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Assumptions:

1. T is an exact copy of S_1 (i.e. fully informative) so $U_2 = 0$.

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0.1	1	1	0	1	$\log(1/3)$	1				
0.4	1	1	1	1	1	1				
Expected values				1	0.610	1	0.390	0	0.610	0

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   - That might make sense with the local interpretation—the only thing unique about S_1 is when it is misinformative.
   - But even this is troublesome as this introduces synergy!!

Final aside: semantics and identity axiom

The 2 bit copy problem

p	s_1	s_2	t
$1/4$	00	0	0
$1/4$	01	0	1
$1/4$	10	1	0
$1/4$	11	1	1

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$1/4$	01	0	1
$1/4$	10	1	0
$1/4$	11	1	1

- The identity axiom is an axiom about a semantic situation

$$R\big(T = (S_1, S_2) : S_1; S_2\big) = MI(S_1, S_2).$$

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The 2 bit copy problem

p	s_1	s_2	t	t'
$1/4$	00	0	0	a
$1/4$	01	0	1	b
$1/4$	10	1	0	c
$1/4$	11	1	1	d

- The identity axiom is an axiom about a semantic situation

$$R\big(T = (S_1, S_2) : S_1; S_2\big) = MI(S_1, S_2).$$

- This is potentially problematic as the same probability distribution may describe two different semantic set-up.

Final aside: semantics and identity axiom

The 2 bit copy problem

p	s_1	s_2	t	t'	t''
$1/4$	00	0	0	a	00
$1/4$	01	0	1	b	11
$1/4$	10	1	0	c	10
$1/4$	11	1	1	d	01

- The identity axiom is an axiom about a semantic situation

$$R\big(T = (S_1, S_2) : S_1; S_2\big) = MI(S_1, S_2).$$

- This is potentially problematic as the same probability distribution may describe two different semantic set-up.

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