# Pointwise Partial Information Decomposition Using Specificity and Ambiguity Lattices

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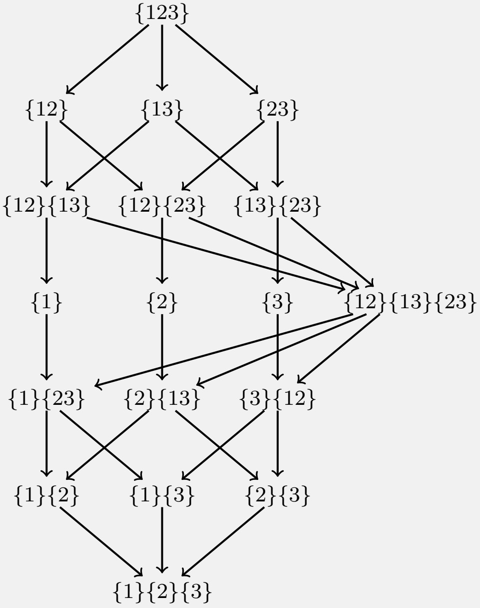
DATA 61 CSIRO



# Unique, redundant and synergistic information

Consider three random variables $S_1$, $S_2$ and $T$

- Aim: predict $T$ using $S_1$ and $S_2$
- Several types of information:

1. **Unique information** $U(S_1 \setminus S_2 \to T)$
2. **Redundant information** $R(S_1, S_2 \to T)$
3. **Complementary information** $C(S_1, S_2 \to T)$

**Unq**

| $p$ | $s_1$ | $s_2$ | $t$ |
|---|---|---|---|
| 1/4 | 0 | 0 | 0 |
| 1/4 | 0 | 1 | 0 |
| 1/4 | 1 | 0 | 1 |
| 1/4 | 1 | 1 | 1 |

**Rdn**

| $p$ | $s_1$ | $s_2$ | $t$ |
|---|---|---|---|
| 1/2 | 0 | 0 | 0 |
| 1/2 | 1 | 1 | 1 |

**Xor**

| $p$ | $s_1$ | $s_2$ | $t$ |
|---|---|---|---|
| 1/4 | 0 | 0 | 0 |
| 1/4 | 0 | 1 | 1 |
| 1/4 | 1 | 0 | 1 |
| 1/4 | 1 | 1 | 0 |

# Bivariate information decomposition

In general, all types of information are present

- Mutual information captures

$$I(S_1;T) = R(S_1,S_2 \to T) + U(S_1 \setminus S_2 \to T)$$

$$I(S_2;T) = R(S_1,S_2 \to T) + U(S_2 \setminus S_1 \to T)$$

**AND**

| $p$ | $s_1$ | $s_2$ | $t$ |
|---|---|---|---|
| 1/4 | 0 | 0 | 0 |
| 1/4 | 0 | 1 | 1 |
| 1/4 | 1 | 0 | 1 |
| 1/4 | 1 | 1 | 1 |

- Joint mutual information captures

$$I(S_{1,2};T) = R(S_1,S_2 \to T) + U(S_1 \setminus S_2 \to T) + U(S_2 \setminus S_1 \to T) + C(S_1,S_2 \to T)$$

- Three equations with four unknowns

# Partial information decomposition

- Axiomatic framework extending this decomposition to arbitrary number of source

**Axioms: PID**

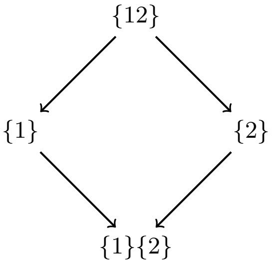
(1) *Symmetry*: $R(S_1, \ldots, S_n \to T)$ is invariant under permutations of the $S_i$'s

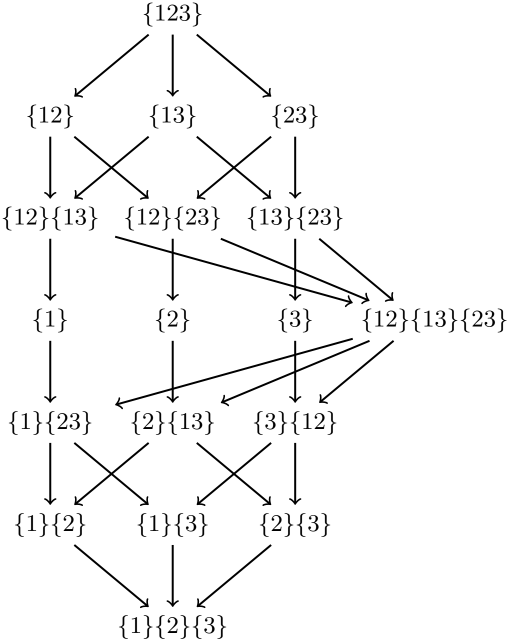
(2) *Monotonicity*: $R(S_1, \ldots, S_n \to T) \leq R(S_1, \ldots, S_{n-1} \to T)$

(3) *Self-redundancy*: $R(S_i \to T) = I(S_i; T)$

- Yields a **redundancy lattice**
- No well-accepted, compatible definition redundant information
- Only consistent for one target variable (no target chain rule)

# Redundancy lattice



# Pointwise information theory

- The mutual information

$$I(X;Y) = \sum_{x\in\mathcal{X},\, y\in\mathcal{Y}} \log \frac{(x,y)}{p(x)p(y)} \geq 0$$

- Woodward (1953) noted the average form of “tempts one to enquire into other simpler methods of derivation [of the per state information]”.

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- Using two postulate, Woodward derived the **pointwise** mutual information

$$i(x;y) = \log \frac{p(x,y)}{p(x)p(y)} \ngeq 0$$

- This was later done more formally by Fano (1961) using four postulates

- Corollaries: (average) mutual information, pointwise entropy and (Shannon) entropy

# Bivariate pointwise information decomposition

- Pointwise decomposition for each realisation

$$\begin{aligned}
i(s_1;t) &= r(s_1, s_2 \to t) + u(s_1 \backslash s_2 \to t) \\
i(s_2;t) &= r(s_1, s_2 \to t) + u(s_2 \backslash s_1 \to t) \\
i(s_{1,2};t) &= r(s_1, s_2 \to t) + u(s_1 \backslash s_2 \to t) + u(s_2 \backslash s_1 \to t) + c(s_1, s_2 \to t)
\end{aligned}$$

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- Should be able to take the expectation over all realisations

$$R(S_1, S_2 \to T) = \big\langle r(s_1, s_2 \to t) \big\rangle \qquad U(S_1 \setminus S_2 \to T) = \big\langle u(s_1 \setminus s_2 \to t) \big\rangle$$
$$C(S_1, S_2 \to T) = \big\langle c(s_1, s_2 \to t) \big\rangle \qquad U(S_2 \setminus S_1 \to T) = \big\langle u(s_2 \setminus s_1 \to t) \big\rangle$$

- This should recover the (average) information decomposition

$$I(S_1;T) = R(S_1, S_2 \to T) + U(S_1 \setminus S_2 \to T)$$
$$I(S_2;T) = R(S_1, S_2 \to T) + U(S_2 \setminus S_1 \to T)$$
$$I(S_{1,2};T) = R(S_1, S_2 \to T) + U(S_1 \setminus S_2 \to T) + U(S_2 \setminus S_1 \to T) + C(S_1, S_2 \to T)$$

# Motivation: PwUnq

- Consider PwUnq from Finn and Lizier (2017b)

| $p$ | $s_1$ | $s_2$ | $t$ | |
|---|---|---|---|---|
| 1/4 | 0 | 1 | 1 | |
| 1/4 | 1 | 0 | 1 | |
| 1/4 | 0 | 2 | 2 | |
| 1/4 | 2 | 0 | 2 | |
| Expected values | | | | |

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| $p$ | $s_1$ | $s_2$ | $t$ | $i_1$ | $i_2$ | $i_{1,2}$ | |
|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 1 | 1 | 0 | 1 | 1 | |
| 1/4 | 1 | 0 | 1 | 1 | 0 | 1 | |
| 1/4 | 0 | 2 | 2 | 0 | 1 | 1 | |
| 1/4 | 2 | 0 | 2 | 1 | 0 | 1 | |
| Expected values | | | | 1/2 | 1/2 | 1 | |

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| $p$ | $s_1$ | $s_2$ | $t$ | $i_1$ | $i_2$ | $i_{1,2}$ | $r$ |
|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1/4 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1/4 | 0 | 2 | 2 | 0 | 1 | 1 | 0 |
| 1/4 | 2 | 0 | 2 | 1 | 0 | 1 | 0 |
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| $p$ | $s_1$ | $s_2$ | $t$ | $i_1$ | $i_2$ | $i_{1,2}$ | $r$ | $u_1$ | $u_2$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1/4 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
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|---|---|---|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1/4 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1/4 | 0 | 2 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1/4 | 2 | 0 | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| Expected values | | | | 1/2 | 1/2 | 1 | 0 | 1/2 | 1/2 | 0 |

- According to $I_{\text{min}}$ Williams and Beer (2010), $\widetilde{UI}$ of Bertschinger et al. (2014), $S_{\text{VK}}$ of Griffith and Koch (2014) and $I_{\text{red}}$ of Harder et al. (2013)

$$R = \langle r \rangle = 1/2 \text{ bit} \neq 0 \text{ bit}$$

- We refer to as the **pointwise unique problem**

# Pointwise partial information decomposition

**Axioms: PPID**

(1) *Symmetry*: $r(s_1, \ldots, s_n \to t)$ is invariant under permutations of the $s_i$'s

(2) *Monotonicity*: $r(s_1, \ldots, s_n \to t) \leq r(s_1, \ldots, s_{n-1} \to t)$

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- Would yield a redundancy lattice for every joint realisation

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- Would yield a redundancy lattice for every joint realisation
- Problems:
    1. Pointwise mutual information is not non-negative
    2. Still no clear definition of redundant information
- PPID needs to overcome this lack of non-negativity

# Probability mass exclusions

- By definition, the pointwise mutual information provided by $s$ about $t$

Surprise of the prior $\quad p(t) \longrightarrow p(t|s) \quad$ Surprise of the posterior

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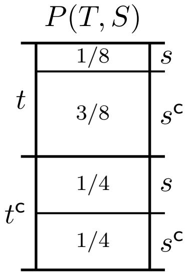

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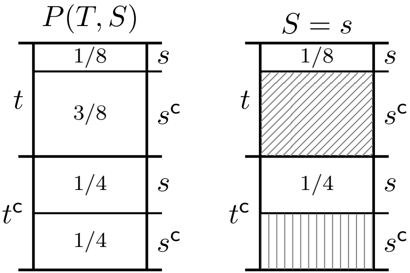

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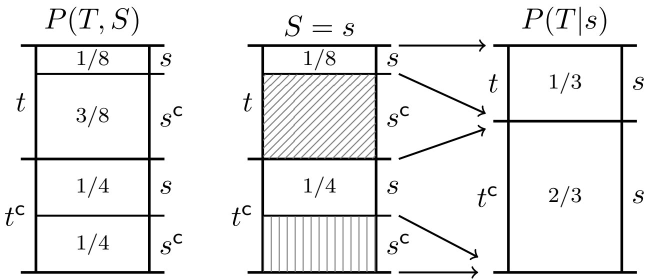
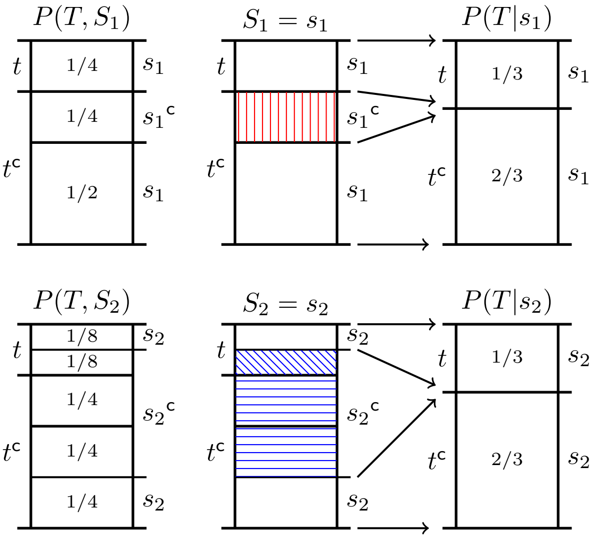


# Motivation for this approach




- The exclusions differ, but yet $i(t; s_1) = i(t; s_2) = {}^4\!/_3$ bit

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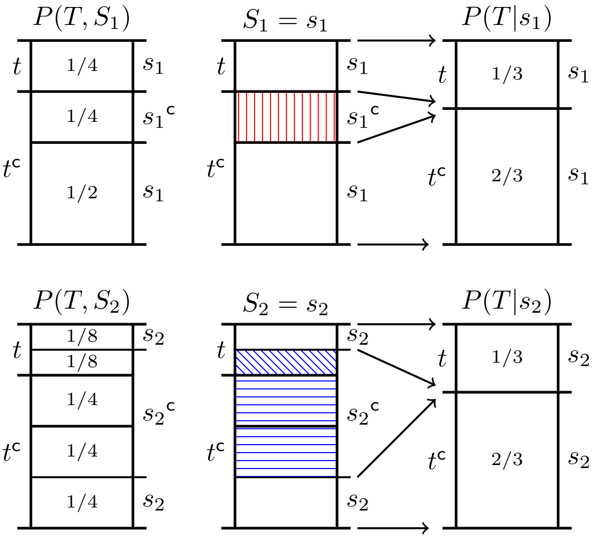
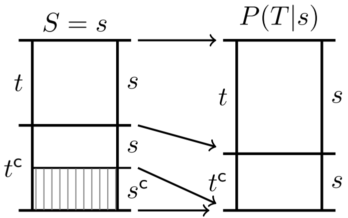
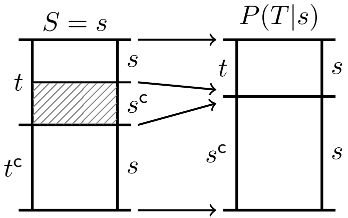



- The exclusions differ, but yet
  $i(t; s_1) = i(t; s_2) = {}^4\!/_3$ bit
- Pointwise MI is not injective
- Same info $\leftrightarrow$ same exclusions

# Two types of exclusions




Purely informative exclusion




Purely misinformative exclusions

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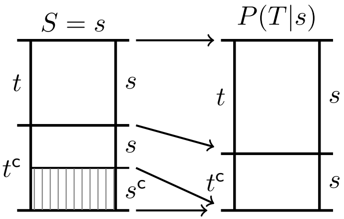
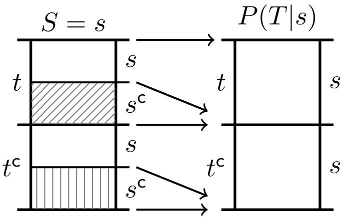
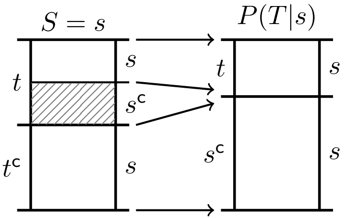



Purely informative exclusion




General case




Purely misinformative exclusions

- Idea: split the pointwise MI into two components

$$i(s \to t) = i^{+}(s \to t) - i^{-}(s \to t)$$

# Postulates for the decomposition

**Postulate 1** The information provided by $s$ about $t$ can be decomposed into two non-negative components,

$$i(s;t) = i^{+}(s \rightarrow t) - i^{-}(s \rightarrow t).$$

**Postulate 2** Each component satisfy a chain rule,

$$i_{+}(s_{1,2} \rightarrow t) = i_{+}(s_1 \rightarrow t) + i_{+}(s_2 \rightarrow t | s_1),$$
$$i_{-}(s_{1,2} \rightarrow t) = i_{-}(s_1 \rightarrow t) + i_{-}(s_2 \rightarrow t | s_1).$$

**Postulate 3** The components $i^{+}(s \rightarrow t)$ and $i^{+}(s \rightarrow t)$ are continuous, monotonically increasing functions the informative and misinformative exclusions, respectively.

# Postulates for the decomposition

**Postulate 1** The information provided by $s$ about $t$ can be decomposed into two non-negative components,

$$i(s;t) = i^+(s \to t) - i^-(s \to t).$$

**Postulate 2** Each component satisfy a chain rule,

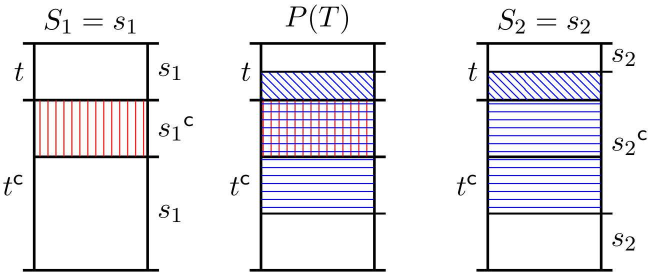
$$i_+(s_{1,2} \to t) = i_+(s_1 \to t) + i_+(s_2 \to t|s_1),$$
$$i_-(s_{1,2} \to t) = i_-(s_1 \to t) + i_-(s_2 \to t|s_1).$$

**Postulate 3** The components $i^+(s \to t)$ and $i^+(s \to t)$ are continuous, monotonically increasing functions the informative and misinformative exclusions, respectively.

- Finn and Lizier (2017a) proved that

  (**Specificity**) $\quad i^+(s \to t) = h(s) \qquad i^-(s \to t) = h(s|t) \quad$ (**Ambiguity**)

# Specificity and ambiguity decomposition




$$i(s_1 \to t) = i(s_2 \to t) = \log {}^4\!/_3 \text{ bit}$$

$i_+(s_1 \to t) = \log {}^4\!/_3$ bit

$i_-(s_1 \to t) = 0$ bit

$i_+(s_2 \to t) = \log \frac{8}{3}$ bit

$i_-(s_2 \to t) = 1$ bit

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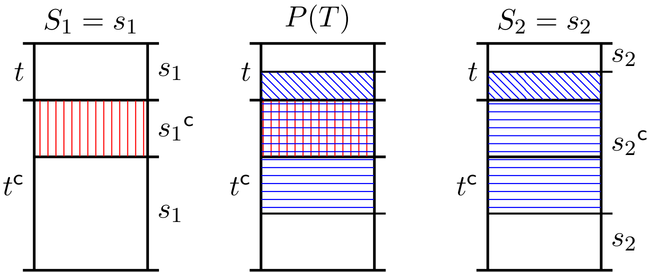


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$i_+(s_2 \to t) = \log \frac{8}{3}$ bit $\qquad$ $i_-(s_2 \to t) = 1$ bit

$r^+(s_1, s_2 \to t) = \log 4/3$ bit, $\qquad u^+(s_2 \backslash s_2 \to t) = \log 4/3$ bit $\qquad u^-(s_2 \backslash s_2 \to t) = \log 1$ bit

# PPID using specificity and ambiguity

- **Finn and Lizier (2017b)** proposes

**Axioms: PPID using Specificity and Ambiguity**

(1) *Symmetry*: $r^{\pm}(s_1, \ldots, s_n \to t)$ is invariant under permutations of the $s_i$'s

(2) *Monotonicity*: $r^{\pm}(s_1, \ldots, s_n \to t) \leq r^{\pm}(s_1; \ldots; s_{n-1} \to t)$

(3) *Self-redundancy*: $r^{\pm}(s_i \to t) = i^{\pm}(s_i; t)$

- Yields a **specificity lattice** and an **ambiguity lattice** for every joint realisation
- Circumvents the non-negativity problem
- Require a measure of redundant specificity and redundant ambiguity

## Pointwise redundant specificity and ambiguity

- The pointwise mutual information $i(s;t)$ does not depend on the apportionment of the probability mass exclusions within the complementary event $t^{\mathsf{c}}$.

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**Axioms: PPID using Specificity and Ambiguity (cont.)**

(4) *Two-event partition*: $r^{\pm}(t : s_1, \ldots, s_n)$ are functions of the probability measures on the two-event partitions $\mathcal{S}_1^{s_1} \times \mathcal{T}^t, \ldots, \mathcal{S}_n^{s_n} \times \mathcal{T}^t$

- Finn and Lizier (2017b) provides arguments justifying this in terms of Kelly gambling

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- Finn and Lizier (2017b) provides arguments justifying this in terms of Kelly gambling
- Leads us to define the redundant specificity and redundant ambiguity

$$r^{+}_{\mathsf{min}}(s_1, \ldots, s_n \to t) = \min_{s_j} h(s_j) \qquad r^{-}_{\mathsf{min}}(s_1, \ldots, s_n \to t) = \min_{s_j} h(s_j|t)$$

- Upon recombination, this measure satisfies the target chain rule

$$r_{\mathsf{min}}\big(s_1, \ldots, s_n \to t_{1,2}\big) = r_{\mathsf{min}}\big(s_1, \ldots, s_n \to t_1\big) + r_{\mathsf{min}}\big(s_1, \ldots, s_n \to t_2|t_1\big),$$

# Bivariate PPID using specificity and ambiguity

- Decomposition of both the specificity and ambiguity the for each realisation

$$
\begin{aligned}
i^{\pm}(s_1;t) &= r^{\pm}(s_1,s_2\!\rightarrow\! t) + u^{\pm}(s_1\backslash s_2\!\rightarrow\! t)\\
i^{\pm}(s_2;t) &= r^{\pm}(s_1,s_2\!\rightarrow\! t) + u^{\pm}(s_2\backslash s_1\!\rightarrow\! t)\\
i^{\pm}(s_{1,2};t) &= r^{\pm}(s_1,s_2\!\rightarrow\! t) + u^{\pm}(s_1\backslash s_2\!\rightarrow\! t) + u^{\pm}(s_2\backslash s_1\!\rightarrow\! t) + c^{\pm}(s_1,s_2\!\rightarrow\! t)
\end{aligned}
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i^{\pm}(s_{1,2};t) &= r^{\pm}(s_1, s_2 \to t) + u^{\pm}(s_1 \setminus s_2 \to t) + u^{\pm}(s_2 \setminus s_1 \to t) + c^{\pm}(s_1, s_2 \to t)
\end{aligned}$$

- Taking the expectation yields

$$R^{\pm}(S_1, S_2 \to T) = \left\langle r^{\pm}(s_1, s_2 \to t) \right\rangle \qquad U^{\pm}(S_1 \setminus S_2 \to T) = \left\langle u^{\pm}(s_1 \setminus s_2 \to t) \right\rangle$$

$$C^{\pm}(S_1, S_2 \to T) = \left\langle c^{\pm}(s_1, s_2 \to t) \right\rangle \qquad U^{\pm}(S_2 \setminus S_1 \to T) = \left\langle u^{\pm}(s_2 \setminus s_1 \to t) \right\rangle$$

- And we have a decomposition of average specificity and ambiguity

$$\begin{aligned}
I^{\pm}(S_1;T) &= R^{\pm}(S_1, S_2 \to T) + U^{\pm}(S_1 \setminus S_2 \to T) \\
I^{\pm}(S_2;T) &= R^{\pm}(S_1, S_2 \to T) + U^{\pm}(S_2 \setminus S_1 \to T) \\
I^{\pm}(S_{1,2};T) &= R^{\pm}(S_1, S_2 \to T) + U^{\pm}(S_1 \setminus S_2 \to T) + U^{\pm}(S_2 \setminus S_1 \to T) + C^{\pm}(S_1, S_2 \to T)
\end{aligned}$$

# Bivariate PPID using specificity and ambiguity (cont.)

- Or recombine the specificity and ambiguity for pointwise information

$$r(s_1, s_2 \to t) = r^+(s_1, s_2 \to t) \; - r^-(s_1, s_2 \to t)$$
$$u(s_1 \setminus s_2 \to t) = u^+(s_1 \setminus s_2 \to t) - u^-(s_1 \setminus s_2 \to t)$$
$$u(s_2 \setminus s_1 \to t) = u^+(s_2 \setminus s_1 \to t) - u^-(s_2 \setminus s_1 \to t)$$
$$c(s_1, s_2 \to t) = c^+(s_1, s_2 \to t) \; - c^-(s_1, s_2 \to t)$$

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- Or recombine the specificity and ambiguity for pointwise information

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\begin{aligned}
r(s_1, s_2 \to t) &= r^+(s_1, s_2 \to t) \; - r^-(s_1, s_2 \to t) \\
u(s_1 \setminus s_2 \to t) &= u^+(s_1 \setminus s_2 \to t) - u^-(s_1 \setminus s_2 \to t) \\
u(s_2 \setminus s_1 \to t) &= u^+(s_2 \setminus s_1 \to t) - u^-(s_2 \setminus s_1 \to t) \\
c(s_1, s_2 \to t) &= c^+(s_1, s_2 \to t) \; - c^-(s_1, s_2 \to t)
\end{aligned}
$$

- Which satisfy the PPID

$$
\begin{aligned}
i(s_1; t) &= r(s_1, s_2 \to t) + u(s_1 \setminus s_2 \to t) \\
i(s_2; t) &= r(s_1, s_2 \to t) + u(s_2 \setminus s_1 \to t) \\
i(s_{1,2}; t) &= r(s_1, s_2 \to t) + u(s_1 \setminus s_2 \to t) + u(s_2 \setminus s_1 \to t) + c(s_1, s_2 \to t)
\end{aligned}
$$

# Bivariate PPID using specificity and ambiguity (cont.)

- Or recombine the specificity and ambiguity for pointwise information

$$
\begin{aligned}
r(s_1, s_2 \to t) &= r^+(s_1, s_2 \to t) - r^-(s_1, s_2 \to t) \\
u(s_1 \setminus s_2 \to t) &= u^+(s_1 \setminus s_2 \to t) - u^-(s_1 \setminus s_2 \to t) \\
u(s_2 \setminus s_1 \to t) &= u^+(s_2 \setminus s_1 \to t) - u^-(s_2 \setminus s_1 \to t) \\
c(s_1, s_2 \to t) &= c^+(s_1, s_2 \to t) - c^-(s_1, s_2 \to t)
\end{aligned}
$$

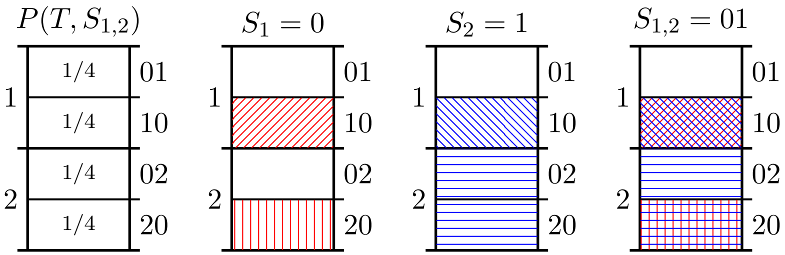
- Which satisfy the PPID

$$
\begin{aligned}
i(s_1; t) &= r(s_1, s_2 \to t) + u(s_1 \setminus s_2 \to t) \\
i(s_2; t) &= r(s_1, s_2 \to t) + u(s_2 \setminus s_1 \to t) \\
i(s_{1,2}; t) &= r(s_1, s_2 \to t) + u(s_1 \setminus s_2 \to t) + u(s_2 \setminus s_1 \to t) + c(s_1, s_2 \to t)
\end{aligned}
$$

- While taking the expectation yields the PID

$$
\begin{aligned}
I(T; S_1) &= R(T : S_1, S_2) + U(T : S_1 \setminus S_2) \\
I(T; S_2) &= R(T : S_1, S_2) + U(T : S_2 \setminus S_1) \\
I(T; S_1 S_2) &= R(T : S_1, S_2) + U(T : S_1 \setminus S_2) + U(T : S_2 \setminus S_1) + C(T : S_1, S_2)
\end{aligned}
$$

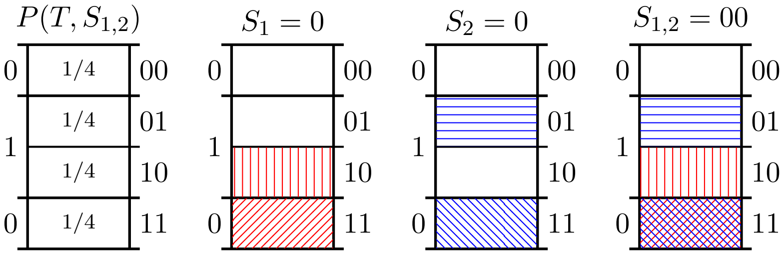
# Example: PwUnq




| $p$ | $s_1$ | $s_2$ | $t$ | $i_1^+$ | $i_1^-$ | $i_2^+$ | $i_2^-$ | $i_{1,2}^+$ | $i_{1,2}^-$ | $r^+$ | $u_1^+$ | $u_2^+$ | $c^+$ | $r^-$ | $u_1^-$ | $u_2^-$ | $c^-$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1/4 | 1 | 0 | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1/4 | 0 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1/4 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| Expected values | | | | 3/2 | 1 | 3/2 | 1 | 2 | 1 | 1 | 1/2 | 1/2 | 0 | 1 | 0 | 0 | 0 |

$R(S_1, S_2 \to T) = 0$ $U(S_1 \backslash S_2 \to T) = 1/2$ $U(S_2 \backslash S_1 \to T) = 1/2$ $C(S_1, S_2 \to T) = 0$

# Example: XOR




| $p$ | $s_1$ | $s_2$ | $t$ | $i_1^+$ | $i_1^-$ | $i_2^+$ | $i_2^-$ | $i_{1,2}^+$ | $i_{1,2}^-$ | $r^+$ | $u_1^+$ | $u_2^+$ | $c^+$ | $r^-$ | $u_1^-$ | $u_2^-$ | $c^-$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1/4 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1/4 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1/4 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| Expected values | | | | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |

$R(S_1, S_2 \to T) = 0$ $\quad U(S_1 \setminus S_2 \to T) = 0$ $\quad U(S_2 \setminus S_1 \to T) = 0$ $\quad C(S_1, S_2 \to T) = 1$

# Comparison to Other Decompositions and Measures

- Similar to Ince (2017) but the monotonicity issue is dealt with in a principled way
- Similar to $I_{\text{min}}$ of Williams and Beer (2010) but now fully pointwise
- Axiom 4 is similar to Assumption $(**)$ of Bertschinger et al. (2014), i.e. measure $\widetilde{UI}$
- This also makes it similar to $S_{\text{VK}}$ of Griffith and Koch (2014)
- There is a target chain rule but no target monotonicity
- There is local positivity on the specificity and ambiguity lattices
- However, upon recombination and taking the expectation, the PI atoms can be negative

# References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Paul L Williams and Randall D Beer. Nonnegative decomposition of multivariate information. *arXiv preprint arXiv:1004.2515*, 2010.

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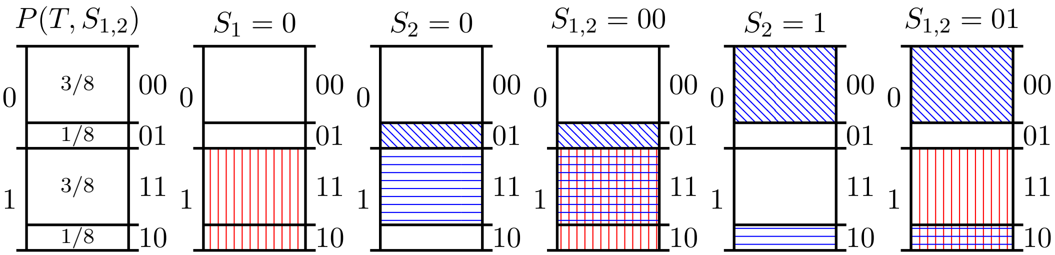
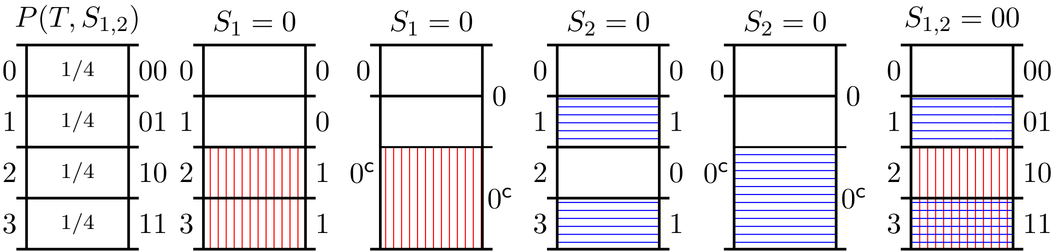
# Example: IMPRDN




| $p$ | $s_1$ | $s_2$ | $t$ | $i_1^+$ | $i_1^-$ | $i_2^+$ | $i_2^-$ | $i_{12}^+$ | $i_{1,2}^-$ | $r^+$ | $u_1^+$ | $u_2^+$ | $c^+$ | $r^-$ | $u_1^-$ | $u_2^-$ | $c^-$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3/8 | 0 | 0 | 0 | 1 | 0 | 1 | $\lg 4/3$ | $\lg 8/3$ | $\lg 4/3$ | 1 | 0 | 0 | $\lg 4/3$ | 0 | 0 | $\lg 4/3$ | 0 |
| 3/8 | 1 | 1 | 1 | 1 | 0 | 1 | $\lg 4/3$ | $\lg 8/3$ | $\lg 4/3$ | 1 | 0 | 0 | $\lg 4/3$ | 0 | 0 | $\lg 4/3$ | 0 |
| 1/8 | 0 | 1 | 0 | 1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 | 0 | 2 | 0 | 0 | 2 | 0 |
| 1/8 | 1 | 0 | 1 | 1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 | 0 | 2 | 0 | 0 | 2 | 0 |
| Expected values | | | | 1 | 0 | 1 | 0.811 | 1.811 | 0.811 | 1 | 0 | 0 | 0.811 | 0 | 0 | 0.811 | 0 |

$R(S_1, S_2 \to T) = 1 \quad U(S_1 \setminus S_2 \to T) = 0 \quad U(S_2 \setminus S_1 \to T) = -0.811 \quad C(S_1, S_2 \to T) = 0.811$

# Example: TwoBitCopy




| $p$ | $s_1$ | $s_2$ | $t$ | $t_{1,2}$ | $t_{1,3}$ | $t_{2,3}$ | $i_1^+$ | $i_1^-$ | $i_2^+$ | $i_2^-$ | $i_{12}^+$ | $i_{12}^-$ | $r^+$ | $u_1^+$ | $u_2^+$ | $c^+$ | $r^-$ | $u_1^-$ | $u_2^-$ | $c^-$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1/4 | 0 | 0 | 0 | 00 | 00 | 00 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1/4 | 0 | 1 | 1 | 01 | 01 | 11 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1/4 | 1 | 0 | 2 | 10 | 11 | 01 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1/4 | 1 | 1 | 3 | 11 | 10 | 10 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Expected values | | | | | | | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

$R(S_1, S_2 \to T) = 1$ $\quad U(S_1 \setminus S_2 \to T) = 0$ $\quad U(S_2 \setminus S_1 \to T) = 0$ $\quad C(S_1, S_2 \to T) = 1$