Redundant information, total information and higher order interactions

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DEMICS – MPI PKS

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Max Planck Institute for Mathematics in the Sciences

MAX-PLANCK-GESELLSCHAFT

Overview

1. The functional form of redundant information

2. The total information lattice

Section 1: The functional form of redundant information

Functional dependence of redundant information

- Does the redundant information depend on the full distribution,

$$I_\cap(T; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k) = f\big(P(T, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)\big)?$$

- Or does it only depend on the marginal distributions,

$$I_\cap(T; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k) = f\big(P(T, \boldsymbol{A}_1), \ldots, P(T, \boldsymbol{A}_k)\big)?$$

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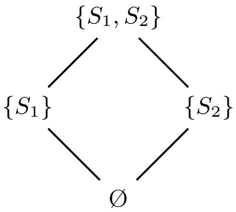

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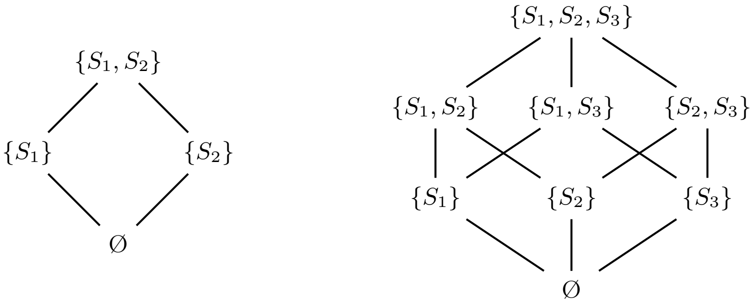

Partial information decomposition

- Mutual information $I(T; \boldsymbol{A})$ quantifies the information provided by a single source
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3. *Monotonicity*:

$$I_\cap(T; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k) \leq I_\cap(T; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_{k-1})$$

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- What are the different ways the sources can provide redundant information?
  - Answering this question corresponds to determining the domain of $I_\cap$

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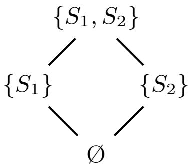

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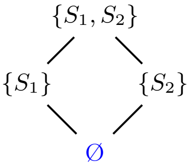

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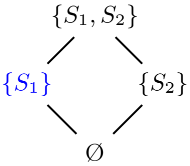

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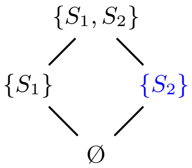

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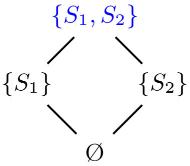

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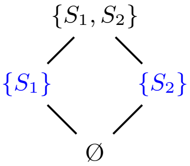

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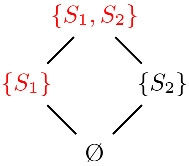

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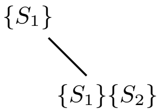

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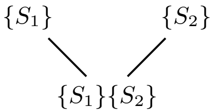

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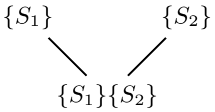

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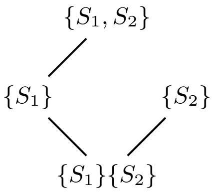

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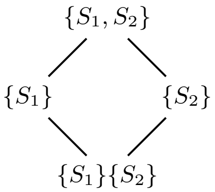

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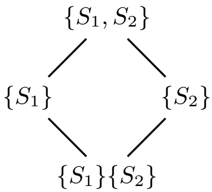

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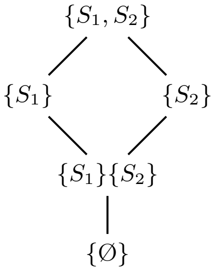

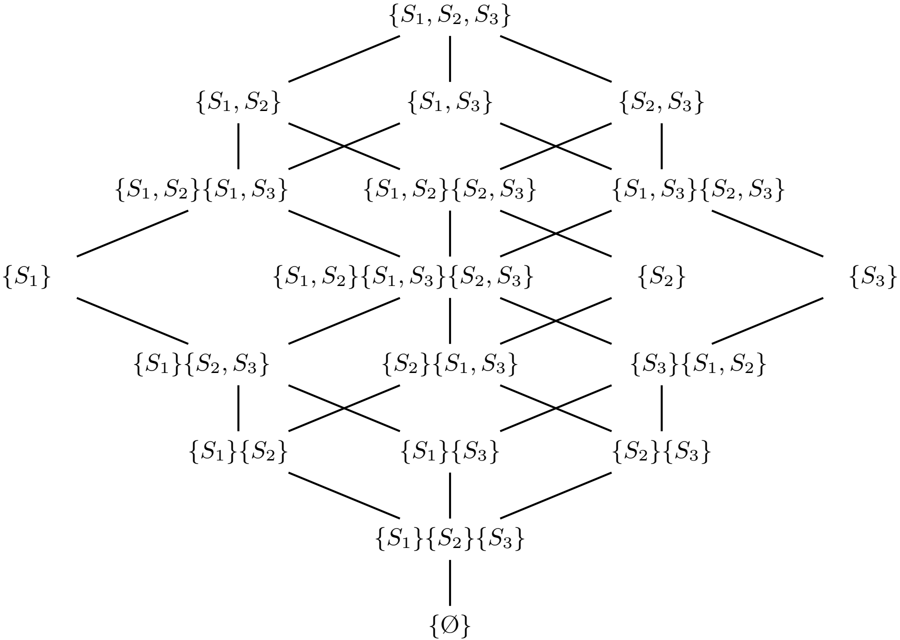
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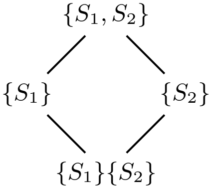

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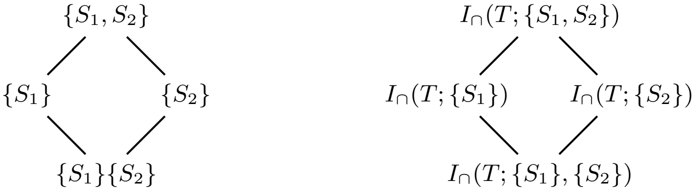

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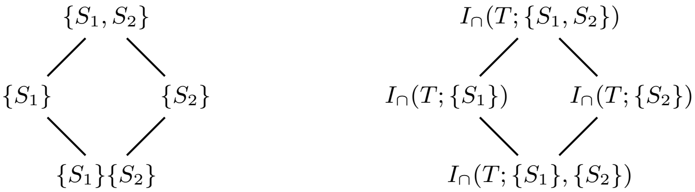

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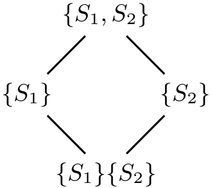

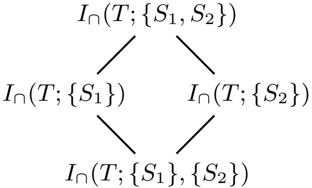

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- Why are we talking about sets of random variables?

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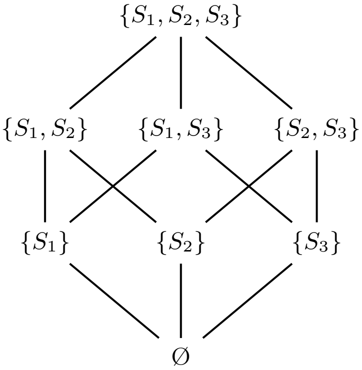

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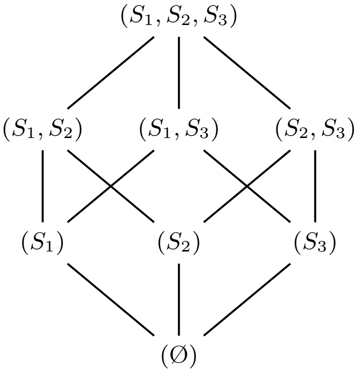

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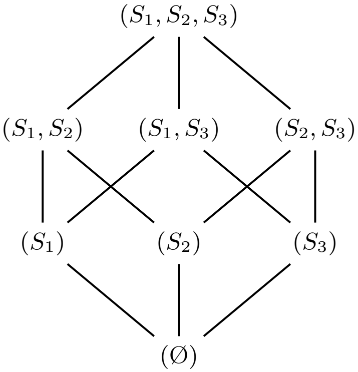

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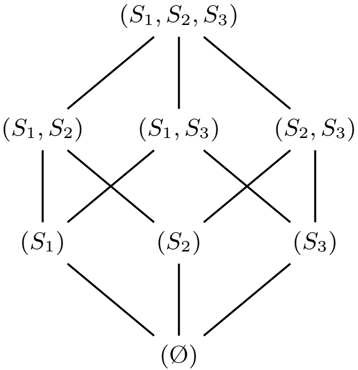

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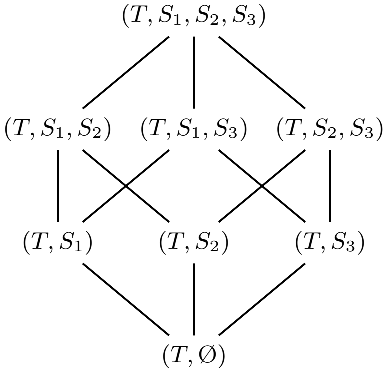

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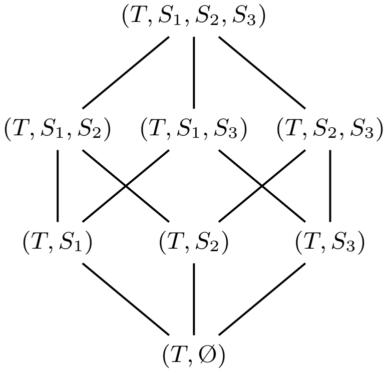
- Lattice of all marginal vectors of $(T, \boldsymbol{S})$ containing T



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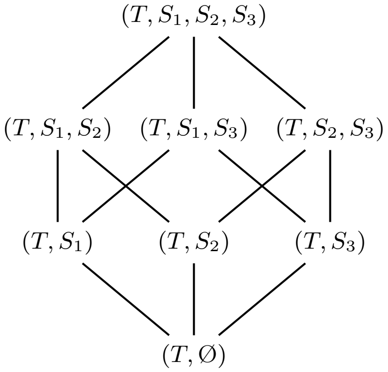
- Lattice of all marginal vectors of $(T, \boldsymbol{S})$ containing T

- Mutual information is a monotonic, bottom normalised lattice function of this lattice



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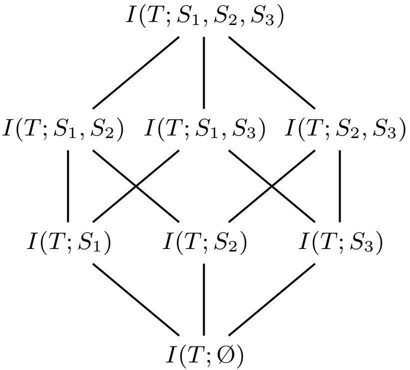

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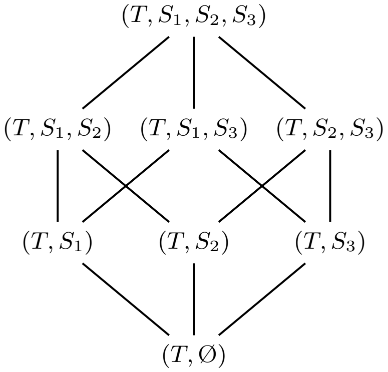

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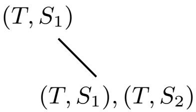

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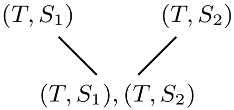

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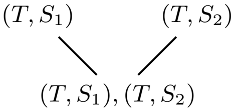

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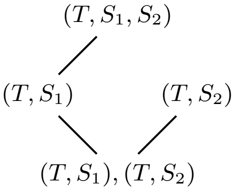

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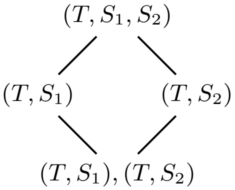

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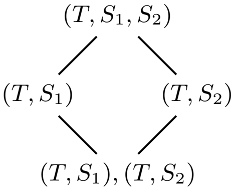

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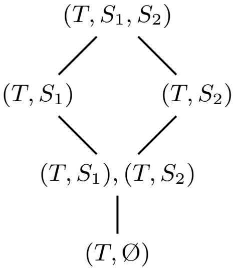

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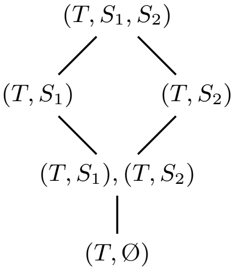

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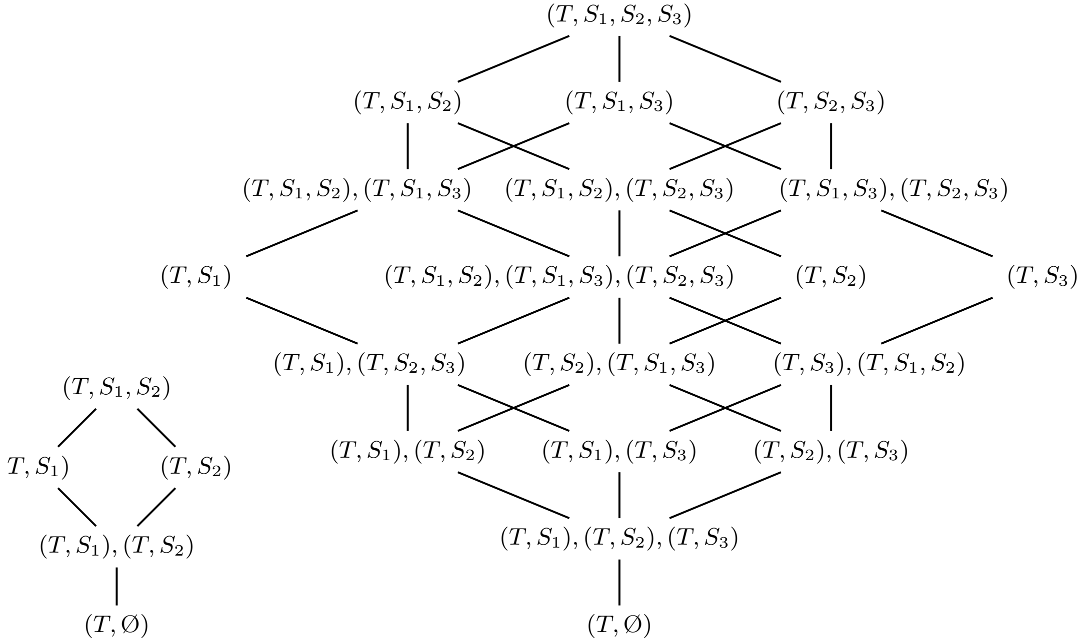

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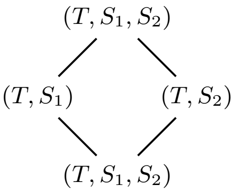

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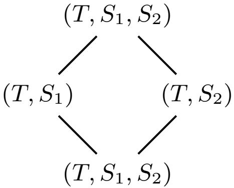

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Section 2: The total information lattice

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  - if we knew the full distribution, then just use the (joint) mutual information

Axioms for total information

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1. *Symmetry*: $I_\cup(T; \boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$ is invariant under permutations of the $\boldsymbol{A}_i$'s
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  - Answering this question corresponds to determining the domain of $I_\cup$

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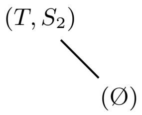

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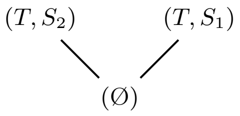

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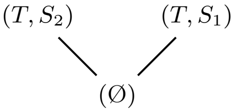

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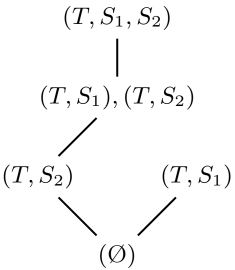

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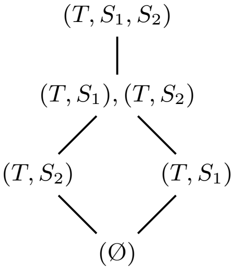

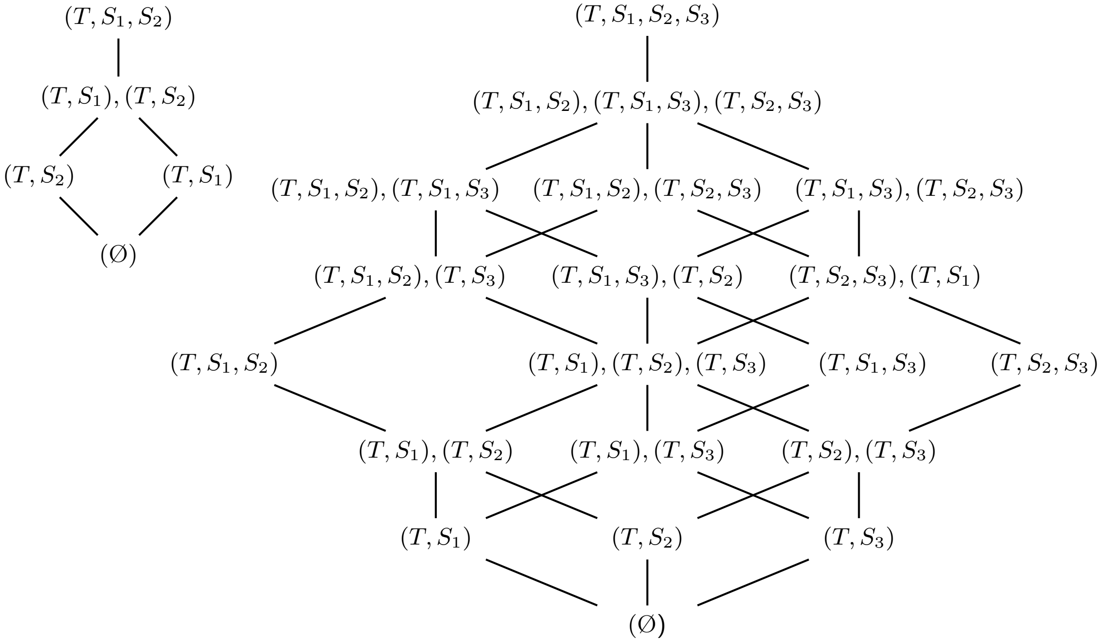

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(∅)

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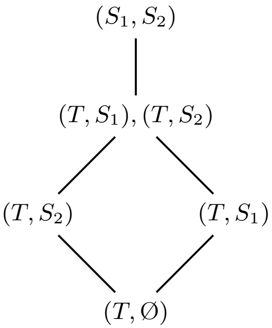

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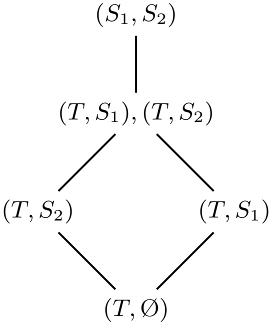

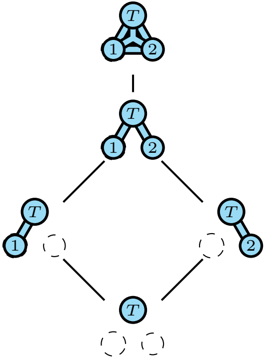

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- The partial information atoms I_∂ are non-negative iff $I_\cap$ is totally monotone

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$$\begin{aligned} I\big(T;(S_1,S_2,S_3)\big) \ge\; & I\big(T;(S_1,S_2)\big) + I\big(T;(S_1,S_3)\big) + I\big((T;(S_2,S_3)\big) \\ & - I_\cap\big(T;(S_1,S_2),(S_1,S_3)\big) - I_\cap\big(T;(S_1,S_2),(S_2,S_3)\big) - I_\cap\big(T;(S_1,S_3),(S_2,S_3)\big) \\ & + I_\cap\big(T;(S_1,S_2),(S_1,S_3),(S_2,S_3)\big) \end{aligned}$$

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- Total monotonicity for $I_\cap$ fails and when a connected $I_\cup$ fails to be monotonic

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– Easier to consider the monotonicity of $I_\cup$ than total monotonicity of $I_\cup$