Two orders for decomposing multivariate information

Conor Finn

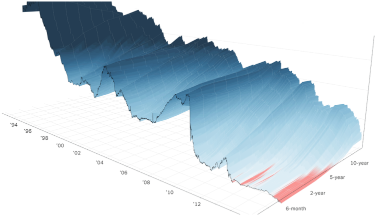
Information Processing in Complex Systems 2022

October 20, 2022

Max Planck Institute for Mathematics in the Sciences

MAX-PLANCK-GESELLSCHAFT

Complex systems



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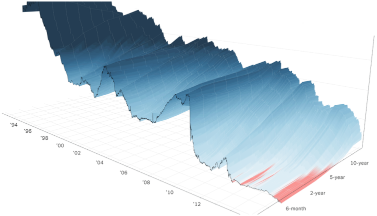

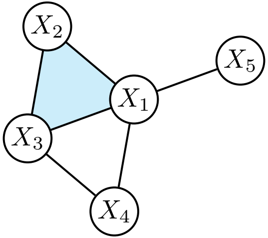

Information theory

- Represent each component using a random variable X_i
- The **entropy** then quantifies our uncertainty about each component

$$H(X_i) = -\sum_{x_i \in X_i} p(x_i) \log p(x_i) \geq 0$$

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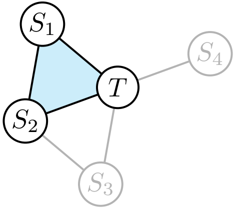
Information theory and higher-order interactions



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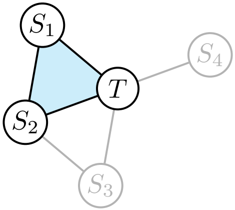

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$\boldsymbol{p}$	$\boldsymbol{s}_1$	$\boldsymbol{s}_2$	$\boldsymbol{t}$
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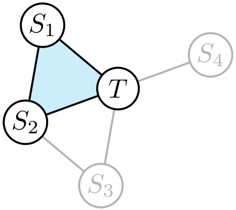

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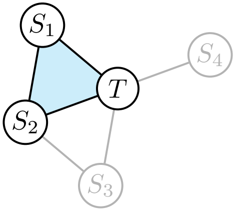

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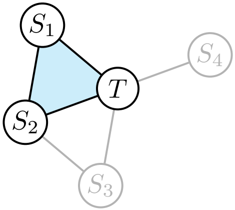

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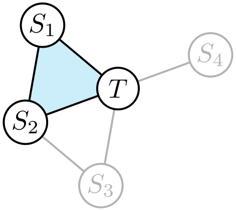

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- Three ways to use the mutual information: $I(S_1;T)$, $I(S_2;T)$ and $I\big((S_1,S_2);T\big)$

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Solution: information decomposition

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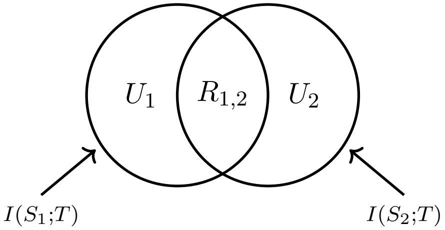

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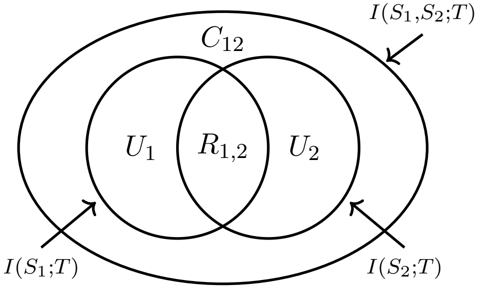

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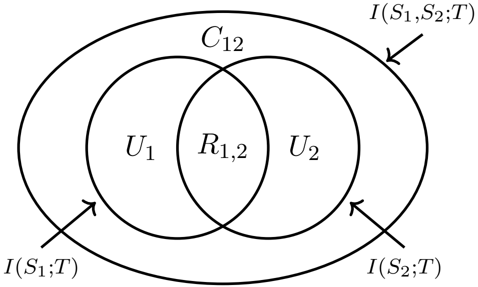

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$$I\big((S_1, S_2);T\big) = U_1 + U_2 + R_{1,2} + C_{12}$$

- Can we define one of the quantities to solve the system?
- How do we generalise this idea to consider more variables?

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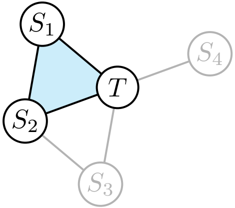

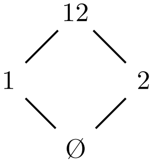

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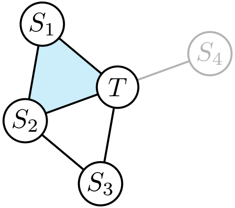

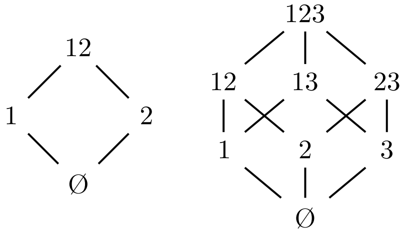

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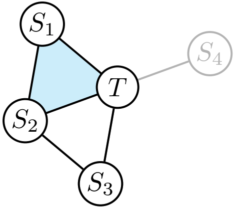

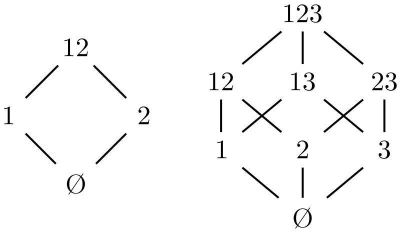

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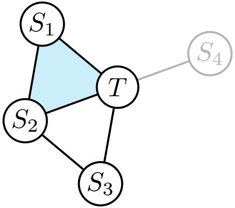

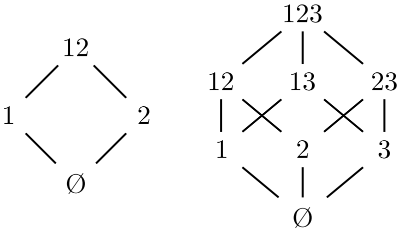

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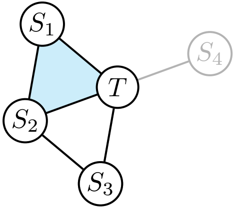

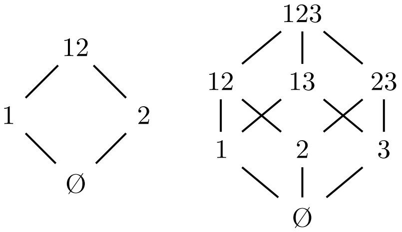

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  - For $n = 2$, we have $\boldsymbol{A}_1 = S_1$, $\boldsymbol{A}_2 = S_2$ and $\boldsymbol{A}_3 = (S_1, S_2)$
- Define a function $I_\cap$ that quantifies the redundant information provided by the $\boldsymbol{A}_i$'s

Redundant information

Williams and Beer axioms

1. *Symmetry*: $I_\cap(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k; T)$ is invariant under permutations of the $\boldsymbol{A}_i$'s
2. *Self-redundancy*: $I_\cap(\boldsymbol{A}_i : T) = I(\boldsymbol{A}_i; T)$
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- Left with all combinations of sources $\boldsymbol{A}_i$ s.t. no source is a subset of any other
  - For $n = 2$, we have

$$I_\cap(\boldsymbol{A}_1; T), \quad I_\cap(\boldsymbol{A}_2; T), \quad I_\cap(\boldsymbol{A}_3; T) \quad \text{and} \quad I_\cap(\boldsymbol{A}_1, \boldsymbol{A}_2; T)$$

- The remaining combinations of sources are the antichains of the inclusion lattice

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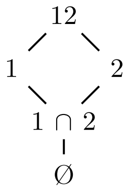
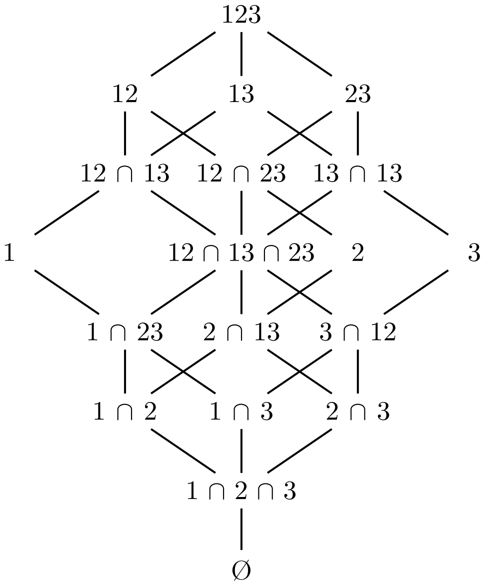
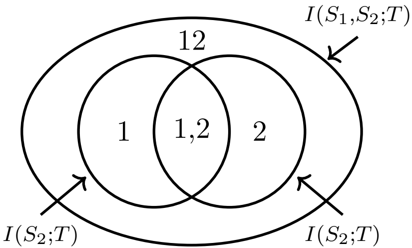
- Applying $\preccurlyeq_\cap$ to all of the combinations of sources yields a lattice structure

Redundancy lattice





Partial information decomposition (PID)



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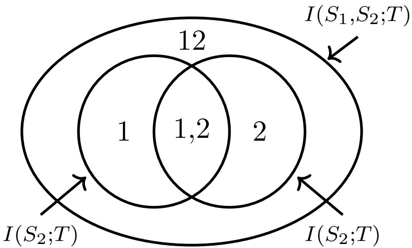

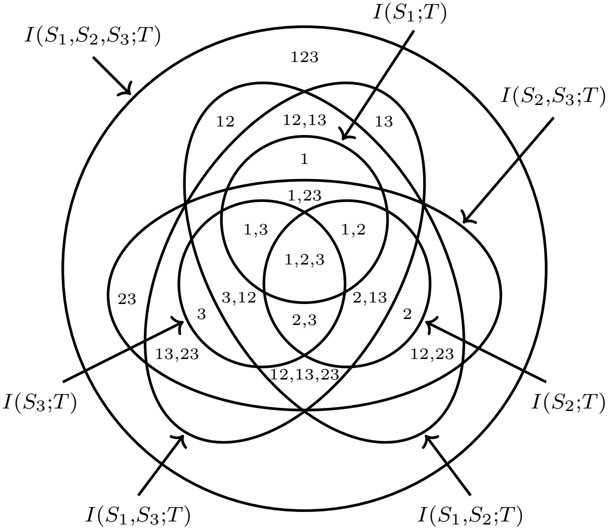

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- Providing this definition has been a contentious area of research
  - Williams and Beer (2010) $I_{\min}$.
  - Harder et al. (2013) I_{red}.
  - Bertschinger et al. (2014) $\widetilde{UI}$, or equivalently Griffith and Koch (2014) S_{VK}.
  - Barrett (2015) I_{MMI}.
  - Finn and Lizier (2018a) $r^{\pm}$.
  - ...

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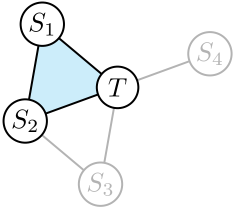

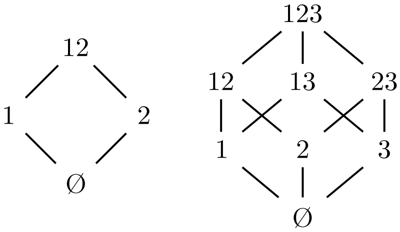

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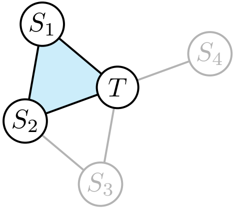
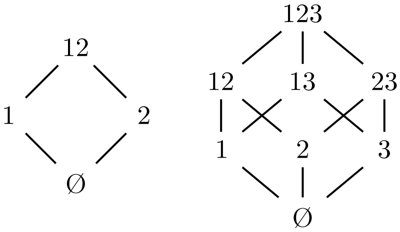
- Williams and Beer focused on defining the redundancy $I_\cap$ between sources
- Can we instead quantify the union information $I_\cup$ provided by sources?





- Let $\boldsymbol{A}_i$ represent a distinct way sources can provide information T
  - For $n = 2$, we have $\boldsymbol{A}_1 = S_1$, $\boldsymbol{A}_2 = S_2$ and $\boldsymbol{A}_3 = (S_1, S_2)$

Union information

Union information axioms

1. *Symmetry*: $I_\cup(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k; T)$ is invariant under permutations of the $\boldsymbol{A}_i$'s
2. *Self-information*: $I_\cup(\boldsymbol{A}_i : T) = I(\boldsymbol{A}_i; T)$
3. *Monotonicity*: $I_\cup(\boldsymbol{A}_1; \ldots; \boldsymbol{A}_k; T) \geq I_\cup(\boldsymbol{A}_1; \ldots; \boldsymbol{A}_{k-1}; T)$ with equality if $\boldsymbol{A}_{k-1} \supseteq \boldsymbol{A}_k$

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- $I_\cup$ can be applied to any combination of $\boldsymbol{A}_i$'s, but many are equivalent
  - For $n = 2$, we have $\boldsymbol{A}_1 = S_1$, $\boldsymbol{A}_2 = S_2$ and $\boldsymbol{A}_3 = (S_1, S_2)$
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- Left with all combinations of sources $\boldsymbol{A}_i$ s.t. no source is a *superset* of any other
  - For $n = 2$, we have

$$I_\cup(\boldsymbol{A}_1; T), \quad I_\cup(\boldsymbol{A}_2; T), \quad I_\cup(\boldsymbol{A}_3; T) \quad \text{and} \quad I_\cup(\boldsymbol{A}_1, \boldsymbol{A}_2; T)$$

- The remaining combinations of sources are the same as for the redundancy

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  - Similarly, $I_\cup(\boldsymbol{A}_3; T) = I_\cup(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3; T) \geq I_\cup(\boldsymbol{A}_1, \boldsymbol{A}_2; T)$

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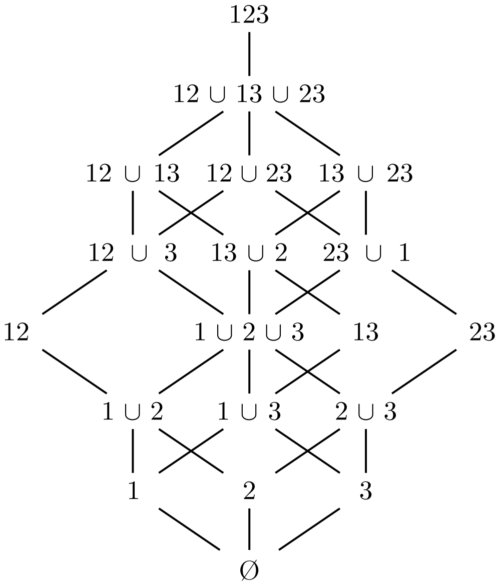
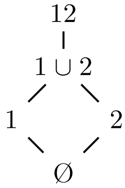
- We have the same combinations of sources, but a different partial order
- Applying $\preccurlyeq_\cup$ to all of the combinations of sources yields a lattice structure

Union information lattice

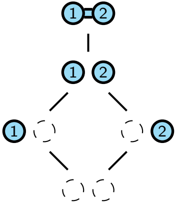


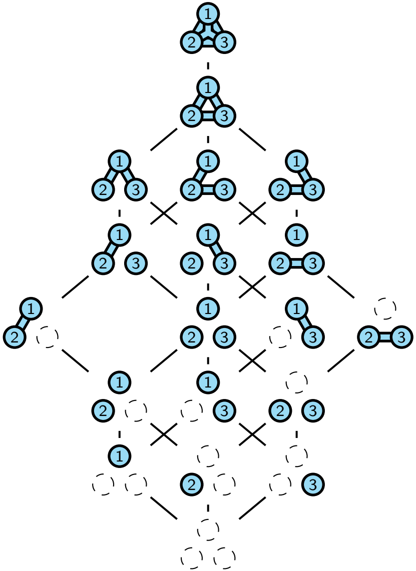


Redundancy and union information

- Redundancy $I_\cap$ and union information $I_\cup$ are dual concepts
- Union information order $\preccurlyeq_\cup$ aligns well with higher-order interactions

Interaction hierarchy





Redundancy and union information

- Redundancy $I_\cap$ and union information $I_\cup$ are dual concepts
- Union information order $\preccurlyeq_\cup$ aligns well with higher-order interactions
- Seems natural to demand a consistency between the approaches

$$I_\cup(S_1, S_2; T) = I(S_1; T) + I(S_2; T) - I_\cap(S_1, S_2; T)$$

- Kolchinsky (2022) argues that we should not make this demand

Inclusion-exclusion principle

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Inclusion-exclusion principle

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  - Williams and Beer (2010) $I_{\min}$, Barrett (2015) I_{MMI}, Finn and Lizier (2018a) $r^{\pm}$

$$
\begin{aligned}
I_{\cap} &= I_{\mathsf{MMI}}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k) = \min\big(I(\boldsymbol{A}_1; T), \ldots, I(\boldsymbol{A}_k; T)\big) \\
\implies I_{\cup} &= I_{\mathsf{MaxMI}} = \max\big(I(\boldsymbol{A}_1; T), \ldots, I(\boldsymbol{A}_k; T)\big)
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\end{aligned}
$$

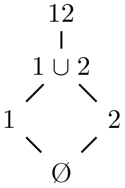
  - Bertschinger et al. (2014) $\widetilde{UI}$, Griffith and Koch (2014) S_{VK}

$$
\begin{aligned}
I_\cap &= \widetilde{SI}(S_1, S_2; T) = \max_{Q \in \Delta P} I_Q(S_1; S_2; T) \\
\implies I_\cup &= \widetilde{UnI}(S_1, S_2; T) = \min_{Q \in \Delta P} I_Q\big((S_1; S_2); T\big)
\end{aligned}
$$

Bounds on the bivariate union information

- Similar to $I_\cap$, the union information $I_\cup$ increases monotonically on the lattice
- In the bivariate case, we have that

$$I\big((S_1,S_2);T\big) \geq I_\cup(S_1,S_2;T) \geq I(S_1;T), I(S_2;T) \geq 0$$



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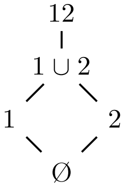

- Assuming that $I_\cup(S_1, S_2; T)$ depends only on $P(S_1, T)$ and $P(S_2, T)$

$$\widetilde{UnI}(S_1, S_2; T) \geq I_\cup(S_1, S_2; T) \geq I_{\mathsf{MaxMI}}$$

Conclusion

Redundant information $I_\cap$ is only one side of the information decomposition problem

We also need to consider the union information $I_\cup$

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Announcement

DeMICS — Decomposing Multivariate Information in Complex Systems

June 5th-9th, 2023

Max Planck Institute for the Physics of Complex systems,
Dresden, Germany

Multivariate mutual information

McGill generalised the MI by defining the **multivariate mutual information**,

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